

Negativity in the Generalized Valence Bond Solid State

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Using a graphical presentation of the spin S one dimensional Valence Bond Solid (VBS) state, based on the representation theory of the $SU(2)$ Lie-algebra of spins, we compute the spectrum of a mixed state reduced density matrix. This mixed state of two blocks of spins A and B is obtained by tracing out the spins outside A and B , in the pure VBS state density matrix. We find in particular that the negativity of the mixed state is non-zero only for adjacent subsystems. The method introduced here can be generalized to the computation of entanglement properties in Levin-Wen models, that possess a similar algebraic structure to the VBS state in the groundstate.

I. INTRODUCTION

The use of quantum information concepts in the study and characterization of many-body systems has proved highly fruitful [1]. A clear example of such concepts is the entanglement entropy, which became an standard tool to characterize gapped, topological and critical phases [2–5]. Despite its usefulness, the entanglement entropy does not fully capture the entanglement in multipartite systems. In systems with more than two components, the state of any two subsystems is described in general by a mixed state. In these systems, a measure of entanglement that generalizes the Perez-Horodecki criterion [6] has been proposed. This measure, called negativity [7] is based on the partial positive transpose (PPT). For a separable state, the partial transpose density matrix (PTDM) is still a density matrix i.e. all its non-vanishing eigenvalues are positive and add to one. In contrast, if a state is not separable, then its PTDM has negative eigenvalues [6]. The sum of the negative eigenvalues of the PTDM is the negativity $\text{Neg}(\rho)$. This quantity is a good measure of entanglement in the sense that it satisfies the following four fundamental criteria [7];

- i) $\text{Neg}(\rho) \geq 0$, being zero just for unentangled states.
- ii) $\text{Neg}(U_{AB}\rho_{AB}U_{AB}^\dagger) = \text{Neg}(\rho)$ for any unitary $U_{AB} = U_A \otimes U_B$.
- iii) $\text{Neg}(\rho)$ does not increase under Local Operations and Classical Communication (LOCC) or post-selection.
- iv) $\text{Neg}(\rho)$ is convex, i.e. for $p_i \leq 0$ and $\sum_i p_i$, $\sum_i p_i \text{Neg}(\rho_i) \geq \text{Neg}(\sum_i p_i \rho_i)$, meaning that it decreases under discarding information.

In systems with a small Hilbert space, the computation of the negativity can be done directly. In many-body systems the sole computation of this entanglement measure becomes challenging. Noteworthy progress has been achieved in systems with bosons [8], free fermions [9, 10], conformally invariant systems in and out of equilibrium [11–13], topological [14, 15] and spin systems [16–18].

In this article we present exact results for the negativity of a series of one dimensional interacting spin S systems, whose groundstate generalize the valence bond solid (VBS) state. The prime example of this type of systems is given by the Affleck, Lieb, Kennedy and Tasaki (AKLT) model. This model represents an example of a gapped spin one system, representative of the Haldane phase [19–21]. In the condensed matter community this model is interesting because it realizes a Symmetry Protected Topological (STP) phase [22, 23]. For open boundary conditions, the system exhibits fractionalized edge modes localized on the edge of the system, and a ground state degeneracy. In the quantum information community, this model has also received attention as a possible resource for measurement based quantum computation [24–26].

The generalized model consists on integer spin S particles arranged on a line. The (frustration free) Hamiltonian is a sum of projectors into total spin $S_{i,i+1}^{\text{tot}} = S_i + S_{i+1} > S$ of consecutive particles. For $S = 1$ it reduces to the usual AKLT model [27]. By tracing subsections of the chain, the reduced density matrix ρ_{AB} of the mixed system of two intervals is obtained. The method presented here to obtain the negativity relies heavily on the representation theory of the $SU(2)$ algebra. We first write the ground state in a matrix product state (MPS) form, in terms of Clebsch-Gordan coefficients. This allows to represent the groundstate partial density matrix as a state sum in a quasi one dimensional lattice, where each link of the lattice has associated a $SU(2)$ representation label. Using F -moves (matrices of change of basis between different but equivalent fusion orderings) and the symmetries of the Clebsch-Gordan coefficients, we can diagonalize ρ_{AB} and its partial transpose $\rho_{AB}^{T_A}$.

This paper is organized as follows. In section II we introduce the generalized spin S VBS model, its MPS representation and its graphical presentation based on the representation theory of the $SU(2)$ Lie algebra. In section III we define the reduced density matrix to study by tracing blocks of spins. We then compute the eigenvalues of such operator by transforming the density matrix into a standard basis in the tensor representation. In section IV we analyze the eigenvalues of the partial transposed density matrix and compute the negativity. Lastly, in

section V we present our conclusions.

II. SPIN S VBS STATE

Let's consider the generalization of the AKLT model for general integer spin S particles, with Hamiltonian

$$\mathcal{H} = \sum_{i=1}^L h_{i,i+1} = \sum_{i=1}^L \sum_{s=S+1}^{2S} \Pi_s(i, i+1), \quad (1)$$

where $\Pi_s(i, i+1)$ is a projector onto the subspace spanned by the s -multiplet formed by spins at i and $i+1$. This Hamiltonian is positive definite. As in the AKLT case, the groundstate can be found exactly by considering each spin S particle as the result of a projection onto the symmetric subspace of two spin $S/2$ particles. Then, each virtual particle of spin $S/2$ is antisymmetrized with its nearest neighbor into the singlet state. The presence of this singlet between consecutive particles prevents the formation of total spin $S_{i,i+1} = S_i + S_{i+1}$ larger than S . Repeating this procedure with every particle in the chain, we obtain a state that is annihilated by the Hamiltonian (1). It corresponds to the groundstate of (1) with eigenvalue zero.

Let us now write down the MPS representation of the spin S VBS state. General boundary conditions can be implemented by boundary tensors u_1, u_L

$$|\mathcal{G}\rangle = u_1^\dagger \mathbf{g}_1 \mathbf{g}_2 \dots \mathbf{g}_L u_L, \quad (2)$$

For a periodic chain of L spins we have

$$|\mathcal{G}\rangle = \text{tr}(\mathbf{g}_1 \mathbf{g}_2 \dots \mathbf{g}_L), \quad (3)$$

where \mathbf{g}_i are $(S+1) \times (S+1)$ matrices. The trace here is done over the auxiliary matrix space. The elements of \mathbf{g}_i and its dual $\bar{\mathbf{g}}_i$ are state vectors:

$$(\mathbf{g}_i)_{ab} = \sum_m \begin{bmatrix} \frac{S}{2} & \frac{S}{2} & S \\ a & -b & m \end{bmatrix} (-1)^b |S, m\rangle_i, \quad (4)$$

$$(\bar{\mathbf{g}}_i)_{ab} = \sum_m \begin{bmatrix} \frac{S}{2} & \frac{S}{2} & S \\ a & -b & m \end{bmatrix} (-1)^b \langle S, m|_i, \quad (5)$$

with $|S, m\rangle$ a state of total spin S and S_z spin projection m . Here $\begin{bmatrix} S_1 & S_2 & J \\ m_1 & m_2 & m \end{bmatrix}$ is the Clebsch-Gordan coefficient of the change of basis between the tensor product of states with definite spin and S_z projection $|S_1, m_1\rangle$ and $|S_2, m_2\rangle$ and the state with definite *total* spin and S_z^{total} projection, i.e.

$$|J, m\rangle = \sum_{m_1, m_2} \begin{bmatrix} S_1 & S_2 & J \\ m_1 & m_2 & m \end{bmatrix} |S_1, m_1\rangle \otimes |S_2, m_2\rangle. \quad (6)$$

The state $|\mathcal{G}\rangle$ is annihilated by the generalized AKLT Hamiltonian (1) with periodic boundary conditions.

Given the symmetry under translations, the ground-state correlation functions are completely determined by

the so called transfer matrix [28]. The transfer matrix for this state is given by $G_{ab}^{cd} = (\bar{\mathbf{g}}_i)_{cd} (\mathbf{g}_i)_{ab}$, taking the inner products of the spin states. Explicitly

$$G_{ab}^{cd} = \sum_m \begin{bmatrix} \frac{S}{2} & \frac{S}{2} & S \\ a & -b & m \end{bmatrix} \begin{bmatrix} \frac{S}{2} & \frac{S}{2} & S \\ c & -d & m \end{bmatrix} (-1)^{S+b+d}. \quad (7)$$

Using the recoupling of Clebsch-Gordan coefficients [28], (see Appendix B), the transfer matrix can be written in the form

$$G_{ab}^{cd} = \sum_{j,m} \lambda_j \begin{bmatrix} \frac{S}{2} & \frac{S}{2} & j \\ -a & c & m \end{bmatrix} \begin{bmatrix} \frac{S}{2} & \frac{S}{2} & j \\ -b & d & m \end{bmatrix} (-1)^{a+b}, \quad (8)$$

where

$$\lambda_j = (-1)^j \frac{(S!)^2 (S+1)}{(S-j)!(S+j+1)!}. \quad (9)$$

Diagrammatic presentation

It is convenient to introduce a diagrammatic presentation of the different tensors appearing above, as otherwise the notation becomes quickly very cumbersome. This presentation also makes clear which manipulations are being done with the different expressions. The building blocks for the diagrammatic presentation are presented in Fig. 1. The concatenation of two objects by joining

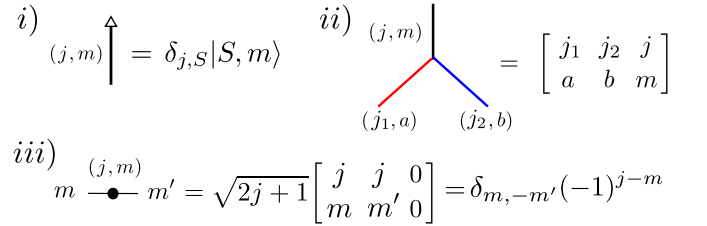


FIG. 1. (color online) i) A north directed arrow represents a ket state vector. An arrow directed south represents a bra (dual state vector). ii) A Clebsch-Gordan coefficient corresponds to a three legged diagram. It is convenient to color the legs of the diagram to keep track of the extra factors between the $3j$ symbols and the Clebsch-Gordan symbols. iii) The diagram with a black dot corresponds to the components of the singlet.

their lines corresponds to sum over all the possible indices of the lines at the concatenation point. The concatenation of a bra and a ket corresponds to taking their inner product.

Using this presentation, the groundstate $|\mathcal{G}\rangle$ is simply

$$|\mathcal{G}\rangle = \begin{matrix} \uparrow & \uparrow & \uparrow & \dots & \uparrow & \uparrow & \uparrow \\ \bullet & \bullet & \bullet & & \bullet & \bullet & \bullet \end{matrix}, \quad (10)$$

where we omit the coloring of the Clebsch-Gordan coefficients as is clear how to contract them. The state $|\mathcal{G}\rangle$ can be thought as a very simple one dimensional lattice

Periodic boundary conditions and the thermodynamic limit

To make further progress, we need to choose some boundary conditions. Assuming periodic boundary conditions (PBC) amounts for contracting the outermost tensors in the diagram above. In the standard basis, after a series of F -moves and recolorings, the reduced density matrix becomes finally

$$\rho_{AB} = \sum_{R=|N_A-N_B|}^{N_A+N_B} \Gamma_{N'_A N'_B}^{N_A N_B}(R) \begin{array}{c} (N_A, m_A) \quad (N_B, m_B) \\ \diagdown \quad \diagup \\ R \\ \diagup \quad \diagdown \\ (N'_A, m'_A) \quad (N'_B, m'_B) \end{array} . \quad (17)$$

An expression of Γ is given in appendix D in terms of $6j$ -symbols. From this expression, the eigenvalues $\Lambda^{(\alpha, R)}$ of the density matrix can be obtained by solving the eigenvalue equation

$$\sum_{n_3 n_4} \Gamma_{n_3 n_4}^{n_1 n_2}(R) e_{n_3 n_4}^{(\alpha, R)} = \Lambda^{(\alpha, R)} e_{n_1 n_2}^{(\alpha, R)} . \quad (18)$$

In the thermodynamic limit of $L_1 + L_3 \rightarrow \infty$, the eigenvalues and eigenvectors of ρ_{AB} can be obtained explicitly as ρ_{AB} becomes

$$\rho_{AB} = \sum_R \Lambda_{N_A N_B}^R \delta_{N'_A}^{N_A} \delta_{N'_B}^{N_B} \begin{array}{c} (N_A, m_A) \quad (N_B, m_B) \\ \diagdown \quad \diagup \\ R \\ \diagup \quad \diagdown \\ (N'_A, m'_A) \quad (N'_B, m'_B) \end{array} , \quad (19)$$

where δ_a^b is a Kronecker delta function. The eigenvalues of ρ_{AB} , labeled by $n, m \in [0, S]$ and $R \in [|n-m|, n+m]$ are

$$\Lambda_{NM}^R = \sum_{j=0}^S (\lambda_j)^{L_2} \left\{ \begin{matrix} M & N & j \\ N & M & R \end{matrix} \right\} \left\{ \begin{matrix} N & N & j \\ \frac{S}{2} & \frac{S}{2} & \frac{S}{2} \end{matrix} \right\} \left\{ \begin{matrix} \frac{S}{2} & \frac{S}{2} & j \\ M & M & \frac{S}{2} \end{matrix} \right\} \\ \times (-1)^{R+j} \frac{(2N+1)(2M+1)\eta_N^{(L_A)}\eta_M^{(L_B)}}{S+1} . \quad (20)$$

Spin $S/2$ at the boundaries

The expression for the reduced density matrix simplifies notably in the case when at the two ends of the chain a spin $S/2$ particle is located. This corresponds to contracting the upper and lower outermost tensor in (16). After some straightforward manipulations we have

$$\rho_{AB} = \sum_{j_2, R} Y_{N'_A N'_B; j_2 R}^{N_A N_B} \begin{array}{c} (N_A, m_A) \quad (N_B, m_B) \\ \diagdown \quad \diagup \\ R \\ \diagup \quad \diagdown \\ (N'_A, m'_A) \quad (N'_B, m'_B) \end{array} . \quad (21)$$

The tensor $Y_{N'_A N'_B; j_2 R}^{N_A N_B}$ is given explicitly in (D1).

IV. NEGATIVITY FOR THE MIXED SYSTEM OF 2 BLOCKS

The negativity of a the subsystem A is defined as the sum of the negative eigenvalues of the PTDM respect to the subsystem A

$$\text{Neg}(\rho_{AB}) = \frac{\|\rho_{AB}^{T_A}\| - 1}{2} = \sum_i \frac{|r_i| - r_i}{2} , \quad (22)$$

with r_i the eigenvalues of $\rho_{AB}^{T_A}$. The norm of the operator $\|O\|$ is defined as $\|O\| = \text{Tr} \sqrt{O^\dagger O}$. Following the same steps that lead us to the density matrix (16), we find

$$\rho_{AB}^{T_A} = \sum_{P, Q} X_{N'_A N'_B; j_1 j_2 j_3}^{N'_A N_B; PQ} \begin{array}{c} (S/2, a') \quad (N'_A, n'_A) \quad (N_B, n_B) \quad (S/2, b') \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ j_1 \quad P \quad j_2 \quad Q \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ (N_A, n_A) \quad (N'_B, n'_B) \end{array} . \quad (23)$$

where the states (N_A, m_A) and (N'_A, m'_A) have been transposed with respect to (16).

The partial transposed density matrix $\rho_{AB}^{T_A}$ (with PBC) is

$$\rho_{AB}^{T_A} = \sum_{R=|N_A-N_B|}^{N_A+N_B} \Gamma_{N'_A N'_B}^{N'_A N_B}(R) \begin{array}{c} (N'_A, m'_A) \quad (N_B, m_B) \\ \diagdown \quad \diagup \\ R \\ \diagup \quad \diagdown \\ (N_A, m_A) \quad (N'_B, m'_B) \end{array} , \quad (24)$$

In the case of two adjacent blocks A and B , where $L_1 = L_2 = L_3$ we find after a straightforward calculation that the operator $\rho_{AB}^{T_A}$ has the simple form

$$U \rho_{AB}^{T_A} U^\dagger = \sum_{N, M} \frac{\sqrt{\eta_N^{(L_A)} \eta_N^{(L_B)} \eta_M^{(L_A)} \eta_M^{(L_B)}}}{\mathcal{N}_{\text{PBC}}} |N, M\rangle \langle M, N| , \quad (25)$$

where $|N\rangle \equiv |N, n\rangle$ is a state of total spin N and S_z projection n . The unitary transformation U is defined by $U|N, m\rangle = (-1)^n |N, -n\rangle$. The normalization for a system of total length L_T and PBC is

$$\mathcal{N}_{\text{PBC}} = \sum_{j=0}^S (2j+1) (\lambda_j)^{L_T} . \quad (26)$$

The eigenvectors and eigenvalues (e, λ) of the operator (25) are

$$\left(|N, n\rangle , \frac{\eta_N^{(L_A)} \eta_N^{(L_B)}}{\mathcal{N}_{\text{PBC}}} \right) , \quad (27)$$

$$\left(\frac{|N, M\rangle \pm |M, N\rangle}{\sqrt{2}} , \pm \frac{\sqrt{\eta_N^{(L_A)} \eta_N^{(L_B)} \eta_M^{(L_A)} \eta_M^{(L_B)}}}{\mathcal{N}_{\text{PBC}}} \right) , \quad (28)$$

where (28) is valid for $N \neq M$ and $n \neq m$. Taking into account the degeneracies of the eigenvalues, we find the negativity to be

$$\text{Neg}(\rho_{AB}) = \mathcal{N}_{\text{PBC}}^{-1} \sum_{N=0}^S (2N+1) \sqrt{\eta_N^{(L_A)} \eta_N^{(L_B)}} \left[N \sqrt{\eta_N^{(L_A)} \eta_N^{(L_B)}} + \sum_{M=N+1}^S (2M+1) \sqrt{\eta_M^{(L_A)} \eta_M^{(L_B)}} \right], \quad (29)$$

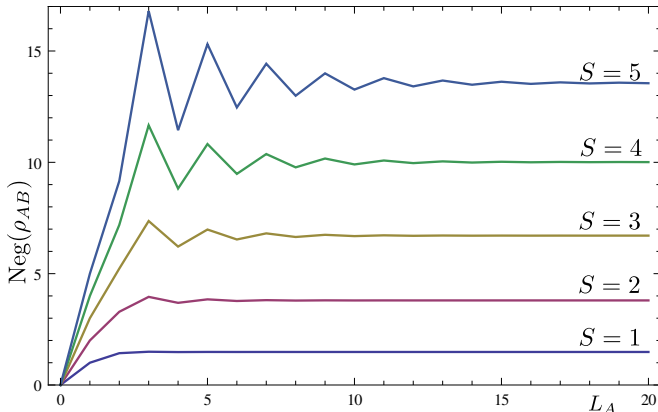


FIG. 2. Negativity of the PTDM as function of block length L_A , for fixed size of block B , ($L_B = 2$). Different curves represent different values of spin S . The continuous line is just a guide to the eye.

valid for a periodic chain of total length $L_T = L_A + L_B$ with $L_{A(B)}$ is the size of the block $A(B)$. For future reference we write explicitly

$$\eta_N^{(L)} = \sum_{j=0}^S (2j+1) (-1)^{N+j+S} (\lambda_j)^L \left\{ \begin{matrix} S/2 & S/2 & j \\ S/2 & S/2 & N \end{matrix} \right\}. \quad (30)$$

with λ_j given by eq. (9). Note that (29) is symmetric under an interchange of A and B , a trivial consequence of $\rho_{AB}^T = \rho_{AB}^T$. Some particular simple cases are

- $L_B = 1$ for any $L_A \rightarrow \text{Neg}(\rho_{AB}) = S$.
- $L_A, L_B \gg 1 \rightarrow \text{Neg}(\rho_{AB}) = \frac{S(S+2)}{2}$.

Other generic cases are plotted in Fig. 2.

General Block separation and LOCC

The negativity found in (29) is valid just for adjacent blocks ($L_1 = L_2 = L_3 = 0$). From the exact density matrix (17) for PBC and (21) for spin $S/2$ particles in the boundary, we can compute the negativity for any fixed value of S, L_A, L_B, L_1, L_2 and L_3 by exact (numerical) diagonalization. In the case of $S = 1, 2, 3$ and PBC we find that whenever $L_1 + L_3 = L_2 = 1$, the negativity vanishes. For general spin S and separations $L_1 + L_3, L_2 \geq 1$, we conjecture the following

- **Conjecture:** The negativity of a bipartite system of two blocks A and B in the one dimensional spin S VBS state vanishes for blocks with no common boundary.

In Ref. [17] we proved this conjecture for spin $S = 1$. Although a proof for a given value of S can be obtained by a direct computation of the negativity for different lengths, for arbitrary value of S this is impracticable. We can further restrict the direct computation to just $L_1 + L_3 = L_2 = 1$ (for spin $S/2$ boundary conditions). As the VBS state can be constructed inductively by series of local operations (projection of two spin $S/2$ particles in each physical site onto the symmetric subspace, plus an antisymmetrization of spin $S/2$ particles on consecutive physical sites), a vanishing negativity $\text{Neg}(\rho_{AB})$ for a particular length ℓ implies that $\text{Neg}(\rho_{AB}) = 0$ for all $L \geq \ell$, due to the monotonicity of the negativity under LOCC. We have indeed verified that this is the case for $S = 2, 3$.

V. CONCLUSION

The AKLT model represents a simple interacting many-body system of interest for both the condensed matter and quantum information community. It realizes the Haldane phase, a gapped phase with symmetry protected topological order. Its VBS groundstate, also serves as a resource for measurement based quantum information. Although being an interacting theory, the VBS groundstate can be written exactly by using simple MPS. The matrices in these MPS are naturally Clebsch-Gordan coefficients between representations of spin S and spin $S/2$ for a chain of spin S particles. Using the representation theory of the $SU(2)$ algebra, we study the bipartite entanglement of the mixed state density matrix obtained by tracing out different sections of the pure state density matrix. In particular, we managed to calculate negativity between two blocks of spins analytically, in the case when the blocks are adjacent. Our results are exact for any value of the spin S as for any length of the blocks, in a chain with periodic boundary conditions. For non adjacent blocks, we checked that the negativity vanishes for $S = 1, 2, 3$, and conjectured that the negativity vanishes for any value of the spin S . We think that vanishing negativity for two non-touching blocks is unique characterization of VBS states. For generic short-range entangled states, we expect the negativity to decay exponentially with the separation between the blocks, with a characteristic decay length proportional to the correlation length. The methods used here allow the computation of multipartite entanglement density matrix, where the $SU(2)$ symmetry is manifest at each stage. We expect that the methods presented here, which rely on the associativity of the tensor product of representations, can be extended to the study of ground state properties in string-net models, where the categorical data in such models satisfies similar

relations to the ones used in this work.

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Appendix A: Identities of Clebsch-Gordan coefficients

The Clebsch-Gordan coefficients satisfy a series of identities. Casted in the diagrammatic presentation, they read

$$\begin{array}{c} (J, m) \\ | \\ \text{---} \text{---} \text{---} \\ / \quad \backslash \\ (j_1, m_1) \quad (j_2, m_2) \end{array} = \begin{array}{c} (J, m) \\ | \\ \text{---} \text{---} \text{---} \\ / \quad \backslash \\ (j_1, m_1) \quad (j_2, m_2) \end{array} = (-1)^{j_1+j_2-J} \begin{array}{c} (J, m) \\ | \\ \text{---} \text{---} \text{---} \\ / \quad \backslash \\ (j_1, m_1) \quad (j_2, m_2) \end{array}, \quad (\text{A1})$$

$$\begin{array}{c} (J, m) \\ | \\ \text{---} \text{---} \text{---} \\ / \quad \backslash \\ (j_1, m_1) \quad (j_2, m_2) \end{array} = (-1)^{J-j_1-j_2} \sqrt{\frac{2J+1}{2j_2+1}} \begin{array}{c} (J, m) \\ | \\ \text{---} \text{---} \text{---} \\ / \quad \backslash \\ (j_1, m_1) \quad (j_2, m_2) \end{array} = (-1)^{J-j_1+j_2} \sqrt{\frac{2J+1}{2j_1+1}} \begin{array}{c} (J, m) \\ | \\ \text{---} \text{---} \text{---} \\ / \quad \backslash \\ (j_1, m_1) \quad (j_2, m_2) \end{array}. \quad (\text{A2})$$

The orthogonality of the Clebsch-Gordan coefficients reads diagrammatically

$$\begin{array}{c} (j_1, m_1) \\ | \\ \text{---} \text{---} \text{---} \\ / \quad \backslash \\ (J, M) \quad (J', M') \\ \backslash \quad / \\ (j_2, m_2) \end{array} = \begin{array}{c} (J, M) \quad (J', M') \\ \text{---} \text{---} \text{---} \end{array} = \delta_{JJ'} \delta_{MM'}. \quad (\text{A3})$$

Appendix B: F -moves

In order to compute the eigenvalues of the transfer matrix \tilde{T} , we make use of the $SU(2)$ structure of \tilde{T} . This matrix corresponds to the contraction of Clebsch-Gordan symbols, so we can use the F -matrix (Racah coefficients [31]) to recouple the coefficients. The recoupling is expressed in Fig. 3

The F -matrix is related to the $6j$ -symbol by

$$F_{NJ}^{J_1 J_2 J_3 J_4} = (-1)^{J_1-2J_3-J_4+N-J} (2J+1) \left\{ \begin{array}{ccc} J_1 & J_2 & N \\ J_4 & J_3 & J \end{array} \right\}. \quad (\text{B1})$$

Appendix C: Orthogonal basis for arbitrary length

The orthogonality of the $[[J, M]_\ell]$ basis can be already verified by manipulating the corresponding diagram where $F_{j'j} \equiv F_{j'j}^{S/2, S/2, S/2, S/2}$ is the F -symbol introduced previously.

Appendix D: Explicit Tensors of reduced density matrix

The X tensor that determines the partial density matrix with general boundary conditions (16) is

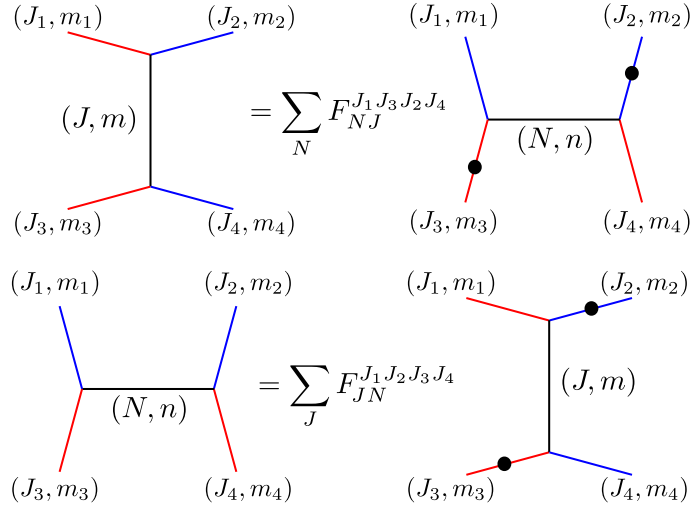


FIG. 3. Recoupling of Clebsch-Gordan coefficients.

$$= \sum_{j=0}^S (\lambda_j)^\ell F_{j'j} = \delta_{JJ'} \delta_{MM'} \sum_{j=0}^S (\lambda_j)^\ell F_{jj}$$

$$X_{n_3 n_4; j_1 j_2 j_3}^{n_1 n_2; PQ} = \mathcal{N} \sqrt{2j_2 + 1} (-1)^{n_1 + n_2 + j_2 + j_3} \prod_{k=1}^4 \frac{\sqrt{(2n_k + 1) \eta_{n_k}}}{S + 1} \prod_{p=1}^3 \sqrt{2j_p + 1} (\lambda_{j_p})^{L_p} F_P^{n_1 j_1 \frac{S}{2} \frac{S}{2}} F_P^{\frac{S}{2} \frac{S}{2} n_3 j_2} F_Q^{n_2 j_3 \frac{S}{2} \frac{S}{2}} F_Q^{\frac{S}{2} \frac{S}{2} n_4 j_2}.$$

The Y tensor that appears in ρ_{AB} with spin $S/2$ boundary conditions (21) is

$$\begin{aligned} Y_{n_3 n_4; j_2 R}^{n_1 n_2} &= (\lambda_{j_2})^{L_2} (-1)^{n_1 + n_3} \prod_{k=1}^4 \sqrt{\frac{(2n_k + 1) \eta_{n_k}}{S + 1}} F_{n_1 \frac{S}{2}}^{\frac{S}{2} \frac{S}{2} n_3 j_2} F_{n_2 \frac{S}{2}}^{\frac{S}{2} \frac{S}{2} n_4 j_2} F_{R j_2}^{n_4 n_3 n_2 n_1}, \\ &= (\lambda_{j_2})^{L_2} (-1)^{n_1 + n_3 + R + j_2} (2j_2 + 1) \left\{ \begin{matrix} n_1 & n_3 & j_2 \\ \frac{S}{2} & \frac{S}{2} & \frac{S}{2} \end{matrix} \right\} \left\{ \begin{matrix} \frac{S}{2} & \frac{S}{2} & j_2 \\ n_2 & n_4 & \frac{S}{2} \end{matrix} \right\} \left\{ \begin{matrix} n_2 & n_4 & j_2 \\ n_3 & n_1 & R \end{matrix} \right\} \prod_{k=1}^4 \sqrt{(2n_k + 1) \eta_{n_k}}. \end{aligned} \quad (D1)$$

The Γ tensor appearing in the partial density matrix with periodic boundary conditions (17) is given by

$$\begin{aligned} \Gamma_{n_3 n_4}^{n_1 n_2}(r) &= \mathcal{N}_{\text{PBC}} (-1)^{n_1 + n_2} \prod_{k=1}^4 \sqrt{(2n_k + 1) \eta_{n_k}} \sum_{p, q, j_1, j_2} (2p + 1)(2q + 1)(2j_1 + 1)(2j_2 + 1) (-1)^{p+q+j_1+j_2} \\ &\quad \times (\lambda_{j_1})^{L_1 + L_3} (\lambda_{j_2})^{L_2} \left\{ \begin{matrix} q & p & r \\ n_3 & n_4 & j_2 \end{matrix} \right\} \left\{ \begin{matrix} n_1 & j_1 & p \\ \frac{S}{2} & \frac{S}{2} & \frac{S}{2} \end{matrix} \right\} \left\{ \begin{matrix} n_2 & j_1 & q \\ \frac{S}{2} & \frac{S}{2} & \frac{S}{2} \end{matrix} \right\} \left\{ \begin{matrix} q & p & r \\ n_1 & n_2 & j_1 \end{matrix} \right\} \left\{ \begin{matrix} n_3 & j_2 & p \\ \frac{S}{2} & \frac{S}{2} & \frac{S}{2} \end{matrix} \right\} \left\{ \begin{matrix} n_4 & j_2 & q \\ \frac{S}{2} & \frac{S}{2} & \frac{S}{2} \end{matrix} \right\}. \end{aligned} \quad (D2)$$

The $\tilde{\Gamma}$ tensor that appears in $\rho_{AB}^{T_A}$ is in turn

$$\begin{aligned} \Gamma_{n_3 n_4}^{n_1 n_2}(r) &= \mathcal{N}_{\text{PBC}} (-1)^{n_1 + n_2} \prod_{k=1}^4 \sqrt{(2n_k + 1) \eta_{n_k}} \sum_{p, q, j_1, j_2} (2p + 1)(2q + 1)(2j_1 + 1)(2j_2 + 1) (-1)^{p+q} (\lambda_{j_1})^{L_1 + L_3} \\ &\quad \times (\lambda_{j_2})^{L_2} \left\{ \begin{matrix} q & p & r \\ n_3 & n_4 & j_2 \end{matrix} \right\} \left\{ \begin{matrix} n_1 & j_1 & p \\ \frac{S}{2} & \frac{S}{2} & \frac{S}{2} \end{matrix} \right\} \left\{ \begin{matrix} n_2 & j_1 & q \\ \frac{S}{2} & \frac{S}{2} & \frac{S}{2} \end{matrix} \right\} \left\{ \begin{matrix} q & p & r \\ n_1 & n_2 & j_1 \end{matrix} \right\} \left\{ \begin{matrix} n_3 & j_2 & p \\ \frac{S}{2} & \frac{S}{2} & \frac{S}{2} \end{matrix} \right\} \left\{ \begin{matrix} n_4 & j_2 & q \\ \frac{S}{2} & \frac{S}{2} & \frac{S}{2} \end{matrix} \right\}. \end{aligned} \quad (D3)$$

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